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The Almost Sure Behavior of Maximal and Minimal Multivariate k_n-Spacings

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Strong limit theorems are obtained for maximal and minimal multivariate k_n-spacings, where $\{k_n\}_{n=1}^{\infty}$ is a sequence of positive integers satisfying $k_n = O(\log n)$. The shapes, in terms of which these spacings are defined, are allowed to be quite general. They must only satisfy certain "entropy" conditions. The main tool for proving our results is a simple relation between these spacings and empirical measures. A number of examples are also included.

1. INTRODUCTION AND MAIN RESULTS

Deheuvels [4] made the "first steps" in the study of the almost sure behavior of multivariate spacings. He defined spacings in terms of multivariate squares. The aim of this paper is to continue the study of

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multivariate spacings. Here, however, we allow the shapes, in terms of which the spacings are defined, to be more general. These are shapes which satisfy certain "entropy" conditions. Since squares in $\mathbb{R}^d$, $d \in \mathbb{N}$, satisfy these conditions, our general definition of spacings in multidimensions includes as a special case the original one used by Deheuvels [4].

Once having defined what we mean by multivariate spacings, we will investigate the almost sure behavior of certain maximal and minimal multivariate spacings. Some of our results for minimal spacings are new even in the classical univariate case. One consequence of our study is that we settle a question raised by Deheuvels [4, p. 423] concerning the behavior of spacings defined in terms of (in our terminology) circles or rectangles, since these shapes satisfy the aforementioned entropy conditions. Essential to our methods is a relation between spacings and empirical measures.

We now specify our setup and introduce some notation. Let $\{X_i\}_{i=1}^{\infty}$ be a sequence of independent uniform $[0, 1]^d$, $d \in \mathbb{N}$, random vectors defined on a probability space $(\Omega, \mathbf{F}, P)$. (See the last two paragraphs of this section for an even more general setup.) We define the empirical measure at stage n by

$$P_n(B) = \frac{1}{n} \sum_{i=1}^{n} 1_B(X_i), \qquad B \in \mathbf{B}^d, \tag{1.1}$$

where $\mathbf{B}^d$ is the class of all Borel measurable subsets of $I^d := [0, 1]^d$. Let $\mathbf{C} \subset \mathbf{B}^d$ be a class of subsets of I^d such that

(C.1) $\phi \in \mathbf{C}$,

(C.2) for all "small" $a > 0$, $\mathbf{C}_a = \{C \in \mathbf{C} : \lambda(C) = a\}$ is non-empty, where λ denotes Lebesgue measure,

(C.3) for all "small" $a > 0$: whenever $C \in \mathbf{C}$ with $\lambda(C) > a$ there exists a $C' \in \mathbf{C}_a$ with $C' \subset C$,

(C.4) for all "small" $a > 0$ and for all $n \in \mathbb{N}$, whenever $\{x_1, ..., x_n\} \subset I^d$ and $C \in \mathbf{C}_a$ there exists a $C' \in \mathbf{C}$ with $\lambda(C') > a$ and

$$\{x_1, ..., x_n\} \cap C = \{x_1, ..., x_n\} \cap C'.$$

Now for any integer $1 \leqslant k < n$ the *maximal k-spacing* (w.r.t. $\mathbf{C}$) at stage n is defined to be

$$M_{k,n} = \sup\{\lambda(C) : C \in \mathbf{C} \text{ and } nP_n(C) < k\}. \tag{1.2}$$

Similarly, let $\mathbf{D} \subset \mathbf{B}^d$ be a class of subsets of I^d such that

(D.1) $I^d \in \mathbf{D}$,

(D.2) for all "small" $a > 0$, $\mathbf{D}_a = \{D \in \mathbf{D} : \lambda(D) = a\}$ is non-empty, (D.3) for all "small" $a>0$: whenever $D \in \mathbf{D}$ with $\lambda(D)<a$ there exists a $D' \in \mathbf{D}_a$ with $D \subset D'$,

(D.4) for all "small" $a>0$ and for all $n \in \mathbb{N}$, whenever $\{x_1, ..., x_n\} \subset I^d$ and $D \in \mathbf{D}_a$ there exists a $D' \in \mathbf{D}$ with $\lambda(D')<a$ and

$$\{x_1, ..., x_n\} \cap D = \{x_1, ..., x_n\} \cap D'.$$

For any integer $1 \leqslant k < n$, the *minimal k-spacing* (w.r.t $\mathbf{D}$) at stage n is defined to be

$$m_{k,n} = \inf\{\lambda(D) \colon D \in \mathbf{D} \text{ and } nP_n(D) > k\}. \tag{1.3}$$

EXAMPLE 1. Let $d=1$ and $\mathbf{C} = \{[a, a+c] \colon 0 \leqslant a \leqslant a+c \leqslant 1\} \cup \{\phi\}$, then

$$M_{k,n} = \max_{0 \leqslant i \leqslant n-k+1} (X_{i+k:n} - X_{i:n}), \tag{1.4}$$

where

$$0 := X_{0:n} \leqslant X_{1:n} \leqslant \cdots \leqslant X_{n:n} \leqslant X_{n+1:n} := 1$$

are the order statistics based on $X_1, ..., X_n$. Similarly, if $\mathbf{D} = \{(a, a+c) \colon 0 \leqslant a < a+c \leqslant 1\} \cup \{I\}$, then

$$m_{k,n} = \min_{1 \leqslant i \leqslant n-k} (X_{i+k:n} - X_{i:n}) \qquad \text{a.s.} \tag{1.5}$$

This example shows that with these choices for $\mathbf{C}$ and $\mathbf{D}$, the definitions of $M_{k,n}$ and $m_{k,n}$ nearly coincide with the classical definitions of maximal and minimal (overlapping) uniform k-spacings. See, e.g., Beirlant and van Zuijlen [2]. The almost sure limiting behavior of these particular spacings has recently been the subject of rather intense investigations. For detailed results for $M_{1,n}$ and $m_{1,n}$, we refer to Slud [21], Devroye [9–11], Deheuvels [3], and Einmahl and van Zuijlen [14], and for results for $M_{k,n}$ when $k>1$ and fixed or when k depends on n ($k=k_n$), we cite Mason [20] and Deheuvels and Devroye [8]. A result for $m_{k,n}$ with k fixed is contained in Beirlant and van Zuijlen [1]. Deheuvels [6] gives a survey of results of this type.

As mentioned above, the first strong limit theorems for multivariate spacings were established in Deheuvels [4]. There $M_{1,n}$ is studied for arbitrary $d \in \mathbb{N}$ and

$$\mathbf{C} = \{[a_1, a_1+c] \times \cdots \times [a_d, a_d+c] \colon 0 \leqslant a_j \leqslant a_j+c \leqslant 1 \text{ for } all\ 1 \leqslant j \leqslant d\} \cup \{\phi\}. \tag{1.6}$$

Further results for $M_{1,n}$ have been obtained by Janson [16].

Note that, in general, it is possible for $M_{k,n}$ and $m_{k,n}$ not to be measurable as functions from Ω to $\mathbb{R}$ and hence not random variables. We circumvent this problem by the following definition: For $A \subset \Omega$, we write A a.s. (almost surely), if there exists an $\Omega_0 \subset A$ with $P(\Omega_0) = 1$.

Of course, the a.s. behavior of $M_{k,n}$ and $m_{k,n}$ depends on how "full" the classes $\mathbf{C}$ and $\mathbf{D}$ are. Therefore, before presenting our main theorems, we must define for a class of sets $\mathbf{E} \subset \mathbf{B}^d$ the following numbers which describe how "full" the class is. For any $0 < a < a(1+\nu) < 1$ write

$$M_{\mathbf{E}}(a, \nu) = \begin{cases} \text{minimum } m \geq 1 & \\ \quad \text{for which there are sets } B_1, \ldots, B_m \in \mathbf{B}^d & \\ \quad \text{such that for any } E \in \mathbf{E}_a = \{E \in \mathbf{E}, \lambda(C) = a\}, & \\ \quad E \subset B_i, \text{ and } \lambda(B_i - E) \leq \nu a \text{ for some } 1 \leq i \leq m; & \\ \infty \quad \text{if no such } m \geq 1 \text{ exists.} \end{cases}$$

For any $0 < (1-\nu)a < a < 1$, write

$$N_{\mathbf{E}}(a, \nu) = \begin{cases} \text{minimum } m \geq 1 & \\ \quad \text{for which there are sets } B_1, \ldots, B_m \in \mathbf{B}^d & \\ \quad \text{such that for any } E \in \mathbf{E}_a,\ B_i \subset E, & \\ \quad \text{and } \lambda(E - B_i) \leq \nu a \text{ for some } 1 \leq i \leq m; & \\ \infty \quad \text{if no such } m \geq 1 \text{ exists.} \end{cases}$$

For any $0 < a < 1$, write

$$K_{\mathbf{E}}(a) = \begin{cases} \text{maximum } m \geq 1 & \\ \quad \text{for which there are sets } E_1, \ldots, E_m \in \mathbf{E}_a & \\ \quad \text{such that } \lambda(E_i \cap E_j) = 0 \text{ for all } i \neq j \ (1 \leq i, j \leq m); & \\ 0 \quad \text{if no such } m \geq 1 \text{ exists.} \end{cases}$$

These numbers are related to the well-known concepts of ε-entropy and ε-capacity. (See, e.g., Kolmogorov and Tihomirov [18] and Gauglhofer and Bharucha-Reid [15].)

Let $\{k_n\}_{n=1}^{\infty}$ be a sequence of integers with $1 \leq k_n < n$. For $\lambda \in (0, \infty)$ let $\alpha_\lambda^+ (\alpha_\lambda^-)$ be the root greater (smaller) than 1 of $x - \log x - 1 = 1/\lambda$.

THEOREM 1. *Assume*

$$\limsup_{a \downarrow 0} \frac{\log N_{\mathbf{C}}(a, \nu)}{\log(1/a)} \leq 1 \qquad \textit{for all small } \nu > 0 \tag{1.7}$$

and

$$\liminf_{a \downarrow 0} \frac{\log K_{\mathbf{C}}(a)}{\log(1/a)} \geq 1. \tag{1.8}$$

(I) *If* $k_n/\log n \to \lambda \in (0, \infty)$, *then*

$$\lim_{n \to \infty} \frac{nM_{k_n,n}}{\log n} = \lambda\alpha_\lambda^+ \qquad \text{a.s.} \tag{1.9}$$

(II) *If* $k_n/\log n \to 0$, *then*

$$\lim_{n \to \infty} \frac{nM_{k_n,n}}{\log n} = 1 \qquad \text{a.s.} \tag{1.10}$$

THEOREM 2. *Assume that condition* (1.7) *holds with* $N_{\mathbf{C}}(a, v)$ *replaced by* $M_{\mathbf{D}}(a, v)$ *and that condition* (1.8) *holds with* $\mathbf{C}$ *replaced by* $\mathbf{D}$.

(I) *If* $k_n/\log n \to \lambda \in (0, \infty)$, *then*

$$\lim_{n \to \infty} \frac{nm_{k_n,n}}{\log n} = \lambda\alpha_\lambda^- \qquad \text{a.s.} \tag{1.11}$$

(II) *If* $k_n = [l_n]$ (*largest integer* $\leqslant l_n$) *for some sequence* $\{l_n\}_{n=1}^{\infty}$ *with* $l_n \uparrow \infty$ *and* $l_n/\log n \downarrow 0$, *then*

$$\lim_{n \to \infty} \frac{k_n \log(nm_{k_n,n}/\log n)}{\log(1/n)} = 1 \qquad \text{a.s.} \tag{1.12}$$

(III) *If* $k \in \mathbb{N}$ *fixed, then*

$$\lim_{n \to \infty} \frac{\log m_{k,n}}{\log(1/n)} = 1 + \frac{1}{k} \qquad \text{a.s.} \tag{1.13}$$

These two theorems give a rather complete description of the a.s. behavior of the maximal and minimal k_n-spacings, when $k_n = O(\log n)$. The next theorem is a generalization of the main result (Theorem 2) in Deheuvels and Devroye [8] and shows what stronger assumptions on $N_{\mathbf{C}}(a, v)$ and $K_{\mathbf{C}}(a)$ can yield.

THEOREM 3. *Assume*

$$\limsup_{a, v \downarrow 0} \frac{\log(aN_{\mathbf{C}}(a, v))}{\log\log(1/a) + \log(1/v)} < \infty \tag{1.14}$$

and

$$\liminf_{a \downarrow 0} \frac{\log(aK_{\mathbf{C}}(a))}{\log\log(1/a)} > -\infty. \tag{1.15}$$

If $k_n = [l_n]^+$ *(smallest integer* $\geqslant l_n$*) for some sequence* $\{l_n\}_{n=1}^{\infty}$ *with* $l_n \uparrow \infty$ *and* $l_n/\log n \downarrow 0$, *then*

$$\lim_{n\to\infty} \frac{nM_{k_n,n} - \log n}{k_n \log((\log n)/k_n)} = 1 \qquad \text{a.s.} \tag{1.16}$$

The remainder of this paper is organized in the following way. First, we provide some examples of classes **C** and **D** which satisfy the conditions on N, M, and K in Theorems 1–3 (the proofs that these classes do indeed satisfy these conditions are deferred to the Appendix) and then we conclude this section by mentioning some possible generalizations of our results. In Section 2, some theorems and inequalities for the increments of the empirical measure are presented and proved. These results, which are likely to be of independent interest, are the main tools for proving Theorems 1 and 2. The proofs of Theorems 1–3 are detailed in Section 3.

The maximal and minimal one-dimensional uniform k-spacings as defined in Example 1 satisfy all the conditions on N, M, and K in Theorems 1–3. This means that our theorems generalize the appropriate parts of the papers cited below Example 1. It should be emphasized, however, that some of these papers give a much more detailed description of the asymptotic behavior of the particular uniform spacings they consider. Theorem 2, with $k > 1$, appears to be new for every possible choice of **D**.

EXAMPLE 2. A more general choice for **C** (**D**) than the one in Example 1, which satisfies all the requisite conditions on N, M, and K in the theorems is the class of all closed (open) rectangles with sides parallel to the coordinate axes, i.e.,

$$\mathbf{C} = \{[a_1, a_1 + c_1] \times \cdots \times [a_d, a_d + c_d] : 0 \leqslant a_j \leqslant a_j + c_j \leqslant 1 \text{ for all } 1 \leqslant j \leqslant d\} \cup \{\phi\} \tag{1.17}$$

and

$$\mathbf{D} = \{(a_1, a_1 + c_1) \times \cdots \times (a_d, a_d + c_d) : 0 \leqslant a_j < a_j + c_j \leqslant 1 \text{ for all } 1 \leqslant j \leqslant d\} \cup \{I^d\}. \tag{1.18}$$

If we replace the rectangles in (1.17) and (1.18) by squares (i.e., we assume $c_1 = c_2 = \cdots = c_d$, see (1.6)) then all the conditions are also satisfied.

EXAMPLE 3. The closed (open) circles are also permissible. To be more precise, let $\|\cdot\|$ denote the Euclidean norm on $\mathbb{R}^d$ and define the closed (open) circles with center $t \in \mathbb{R}^d$ by

$$\bar{B}_r(t) = \{s \in \mathbb{R}^d : \|s - t\| \leqslant r\}, \qquad r \geqslant 0, \tag{1.19}$$

and

$$B_r(t) = \{s \in \mathbb{R}^d: \|s - t\| < r\}, \qquad r > 0, \tag{1.20}$$

Now write

$$\mathbf{C} = \{\bar{B}_r(t) \subset I^d: r \geqslant 0, t \in I^d\}, \tag{1.21}$$

$$\mathbf{D} = \{B_r(t) \subset I^d: r > 0, t \in I^d\}, \tag{1.22}$$

then $\mathbf{C}$ and $\mathbf{D}$ satisfy all the "entropy" conditions of Theorem 1–3. If the Euclidean norm is replaced by the maximum-norm in (1.19) and (1.20), then $\mathbf{C}$ and $\mathbf{D}$ in (1.21) and (1.22) are just the squares mentioned below (1.18). More generally, it can be shown, using a result in Kolmogorov and Tihomirov [18, p. 296], that $\mathbf{C}$ and $\mathbf{D}$ in (1.21) and (1.22) satisfy the conditions of Theorems 1 and 2 if $\|\cdot\|$ is *any norm* on $\mathbb{R}^d$.

These last two examples answer to the affirmative the question posed by Deheuvels [4] mentioned above.

The setup of this paper can be extended in a number of ways. Instead of having the uniform distribution on I^d, we could have assumed that the X_i are distributed according to a distribution function F having a continuous density f. (Note, however, that we would maintain the Lebesgue measure λ in the assumptions on $\mathbf{C}$ and $\mathbf{D}$.) Under this last assumption, some results for $M_{1,n}$, when $d = 1$, have been obtained by Deheuvels [5, 7]. The theorems in this paper can easily be generalized in this direction using the ideas in Einmahl [12, Section 6.2]. Another avenue towards generalization is to replace I^d by an arbitrary (closed, bounded) subset $A \subset \mathbb{R}^d$ (with non-empty interior). It is also easy to modify the results and proofs to this situation.

2. Results on the Increments of the Empirical Measure

Recall the classes $\mathbf{C}$ and $\mathbf{D}$ used in the definitions of $M_{k,n}$ and $m_{k,n}$, respectively. Define for "small" $a > 0$

$$\Delta_n^+(a) = \max\{nP_n(E): E \in \mathbf{D}_a\} \tag{2.1}$$

and

$$\Delta_n^-(a) = \min\{nP_n(E): E \in \mathbf{C}_a\}. \tag{2.2}$$

The purpose of this section is to establish results on the a.s. behavior of Δ_n^+ and Δ_n^- when $a = a_n = O((\log n)/n)$. So let $\{a_n\}_{n=1}^\infty$ be a sequence of such numbers with $0 < a_n < 1$. Furthermore, for $c > 0$ let β_c^+ be the root > 1 of

$\beta(\log\beta - 1) + 1 = 1/c$ and $c > 1$ let β_c^- be the root < 1 of the same equation and set $\beta_c^- = 0$ for $0 < c \leqslant 1$.

THEOREM 4. *Assume*

$$\limsup_{a \downarrow 0} \frac{\log M_D(a, v)}{\log(1/a)} \leqslant 1 \qquad \text{for all small } v > 0, \tag{2.3}$$

$$\limsup_{a \downarrow 0} \frac{\log N_C(a, v)}{\log(1/a)} \leqslant 1 \qquad \text{for all small } v > 0 \tag{2.4}$$

and

$$\liminf_{a \downarrow 0} \frac{\log K_{\mathbf{E}}(a)}{\log(1/a)} \geqslant 1 \qquad \text{for } \mathbf{E} = \mathbf{C} \text{ and for } \mathbf{E} = \mathbf{D}. \tag{2.5}$$

(I) *If* $a_n = (c \log n)/n$ $(c \in (0, \infty))$, *then for either choice of sign*

$$\lim_{n \to \infty} \frac{\Delta_n^{\pm}(a_n)}{\log n} = c\beta_c^{\pm} \qquad \text{a.s.,} \tag{2.6}$$

if $c < 1$, *then even*

$$\lim_{n \to \infty} \Delta_n^-(a_n) = 0 \qquad \text{a.s.} \tag{2.7}$$

(II) *Write* $c_n = (na_n)/\log n$. *If* $c_n \downarrow 0$ *and* $\log(1/c_n)/\log n \downarrow 0$, *then*

$$\lim_{n \to \infty} \frac{\log(1/c_n)}{\log n} \Delta_n^+(a_n) = 1 \qquad \text{a.s.} \tag{2.8}$$

(III) *If* $a_n = n^{-c}$ $(c \in (1, \infty))$ *and* $c/(c-1) \notin \mathbb{N}$, *then*

$$\lim_{n \to \infty} \Delta_n^+(a_n) = [c/(c-1)] \qquad \text{a.s.} \tag{2.9}$$

For a complete description of the a.s. behavior of $\Delta_n^{\pm}$ when $\mathbf{C}$ and $\mathbf{D}$ are defined as in (1.17) and (1.18) (i.e., classes of rectangles), we refer to Einmahl and Ruymgaart [13] or Einmahl [12, Chaps. 5 and 6]. There, sequences which decrease slower than $(c \log n)/n$ are also considered.

Before we proceed to the proof of Theorem 4, we first present a number of inequalities which we will need there. We begin with a well-known fact for binomial random variables. Define $h(x) = x(\log x - 1) + 1$, $x > 0$.

FACT 1. Let $n \in \mathbb{N}$, $0 < p < 1$, and $B(n, p)$ a binomial random variable with parameters n and p. Then we have for $\lambda > 1$

$$P(B(n, p) \geqslant \lambda np) \leqslant \exp(-nph(\lambda)) \tag{2.10}$$

and

$$P(B(n, p) \leqslant np/\lambda) \leqslant \exp(-nph(1/\lambda)). \tag{2.11}$$

FACT 2 (Mallows [19]). If $\langle N_1, ..., N_m \rangle$, $m \in \mathbb{N}$, is multinomially distributed with parameters n and $p_1, ..., p_m$, where $n \in \mathbb{N}$ and $p_1, ..., p_m$ are nonnegative with $\sum_{j=1}^{m} p_j = 1$, then we have for any $\lambda \geqslant 0$

$$P(\max_{1 \leqslant j \leqslant m} N_j \leqslant \lambda) \leqslant \prod_{j=1}^{m} P(N_j \leqslant \lambda) \tag{2.12}$$

and

$$P(\min_{1 \leqslant j \leqslant m} N_j \geqslant \lambda) \leqslant \prod_{j=1}^{m} P(N_j \geqslant \lambda). \tag{2.13}$$

Since $\Delta_n^+(a)$ and $\Delta_n^-(a)$ need not be measurable, we must use the outer probability measure, denoted by P^*, in our inequalities for these quantities.

INEQUALITY 1. *For any "small" $a > 0$ and v, λ such that $a < a(1+v) < 1$ and $\lambda > 1 + v$ we have*

$$P^*(\Delta_n^+(a) \geqslant \lambda na) \leqslant M_{\mathrm{D}}(a, v) \exp\{-na(1+v)\, h(\lambda/(1+v))\}; \tag{2.14}$$

for any "small" $a > 0$ and v, λ such that $0 < a(1-v) < a$ and $\lambda > 1$ we have

$$P^*(\Delta_n^-(a) \leqslant na/\lambda) \leqslant N_{\mathrm{C}}(a, v) \exp\{-na(1-v)\, h(1/(\lambda(1-v)))\}. \tag{2.15}$$

Proof. These inequalities follow readily using the sets $B_1, ..., B_m$, introduced in the definitions of $M_{\mathrm{D}}(a, v)$ and $N_{\mathrm{C}}(a, v)$, and applying the inequalities in Fact 1. ∎

INEQUALITY 2. *For any "small" $a > 0$ and $\lambda \geqslant 0$, we have*

$$P^*(\Delta_n^+(a) < \lambda) \leqslant \exp\{-K_{\mathrm{D}}(a)\, P(B(n, a) \geqslant \lambda)\} \tag{2.16}$$

and

$$P^*(\Delta_n^-(a) > \lambda) \leqslant \exp\{-K_{\mathrm{C}}(a)\, P(B(n, a) \leqslant \lambda)\}. \tag{2.17}$$

Proof. We only prove (2.16), since the proof of (2.17) is nearly the same. Let $\mathbf{P}_a$ be a set $\{E_1, ..., E_m\}$ in the definition of $K_{\mathrm{D}}(a)$ with $\#\mathbf{P}_a = K_{\mathrm{D}}(a)$. Then we have, using (2.12),

$$\begin{aligned} P^*(\Delta_n^+(a) < \lambda) &\leqslant P(\max_{E \in \mathbf{P}_a} nP_n(E) < \lambda) \\ &\leqslant \prod_{E \in \mathbf{P}_a} P(nP_n(E) < \lambda) = P(B(n, a) < \lambda)^{K_{\mathrm{D}}(a)} \\ &= (1 - P(B(n, a) \geqslant \lambda))^{K_{\mathrm{D}}(a)} \\ &\leqslant \exp\{-K_{\mathrm{D}}(a)\, P(B(n, a) \geqslant \lambda)\}. \quad ∎ \end{aligned}$$

In the proof of Theorem 4, parts I and II, we need the following fact in order to evaluate the binomial probabilities in (2.16) and (2.17).

FACT 3 (Kiefer [17]). Assume $n^{1/2}a_n \to 0$, $\lambda_n \to \infty$, and $n^{-1/2}\lambda_n \to 0$. If $\limsup_{n\to\infty} na_n/\lambda_n < 1$, then

$$\log P(B(n, a_n) \geqslant \lambda_n) = \lambda_n\{\log(na_n/\lambda_n) - na_n/\lambda_n + 1 + o(1)\}; \quad (2.18)$$

if $\liminf_{n\to\infty} na_n/\lambda_n > 1$, then

$$\log P(B(n, a_n) \leqslant \lambda_n) = \lambda_n\{\log(na_n/\lambda_n) - na_n/\lambda_n + 1 + o(1)\}. \quad (2.19)$$

Proof of Theorem 4. The proofs of (2.6)–(2.9) are all very similar. Therefore, we restrict ourselves for the sake of brevity to a proof of (2.6) for Δ_n^+ and a proof of (2.8). See also Einmahl and Ruymgaart [13] for a detailed proof that $[c/(c-1)]$ is a lower bound for the lim sup in part (III), when **D** is the class of rectangles.

We first prove that

$$\limsup_{n\to\infty} \Delta_n^+(a_n)/\log n \leqslant c\beta_c^+ \qquad \text{a.s.} \quad (2.20)$$

if $a_n = (c \log n)/n$. Let $\varepsilon > 0$ arbitrary (but "small"), and write $n_k = [(1+\varepsilon/2)^k]$ for $k \in \mathbb{N}$. Observe that

$$\max_{n_{k-1} < n \leqslant n_k} \Delta_n^+(a_n)/\log n \leqslant \Delta_{n_k}^+(a_{n_{k-1}})/\log n_{k-1}. \quad (2.21)$$

Hence by the Borel–Cantelli lemma, it suffices to show that $\sum p_k < \infty$, with $p_k = P^*(\Delta_{n_k}^+(a_{n_{k-1}}) \geqslant (1+2\varepsilon)\, c\beta_c^+ \log n_{k-1})$. Using (2.14), we have for large k

$$\begin{aligned} p_k &= P^*(\Delta_{n_k}^+(a_{n_{k-1}}) \geqslant (1+2\varepsilon)\, \beta_c^+ n_k a_{n_{k-1}} n_{k-1}/n_k) \\ &\leqslant P^*(\Delta_{n_k}^+(a_{n_{k-1}}) \geqslant (1+\varepsilon)\, \beta_c^+ n_k a_{n_{k-1}}) \\ &\leqslant M_{\mathbf{D}}(a_{n_{k-1}}, \varepsilon) \exp\{-n_k a_{n_{k-1}}(1+\varepsilon)\, h(\beta_c^+)\} \\ &= M_{\mathbf{D}}(a_{n_{k-1}}, \varepsilon) \exp\{-n_k(1+\varepsilon)(\log n_{k-1})/n_{k-1}\} \\ &\leqslant M_{\mathbf{D}}(a_{n_{k-1}}, \varepsilon)\, n_{k-1}^{-(1+\varepsilon)}. \end{aligned} \quad (2.22)$$

From (2.22) and (2.3), we have

$$\begin{aligned} \limsup_{k\to\infty} \frac{\log p_k}{\log n_{k-1}} &\leqslant \limsup_{k\to\infty} \frac{\log M_{\mathbf{D}}(a_{n_{k-1}}, \varepsilon) - (1+\varepsilon)\log n_{k-1}}{\log n_{k-1}} \\ &= \left(\limsup_{k\to\infty} \frac{\log M_{\mathbf{D}}(a_{n_{k-1}}, \varepsilon)}{\log(1/a_{n_{k-1}})}\right) - (1+\varepsilon) \\ &\leqslant 1 - (1+\varepsilon) = -\varepsilon. \end{aligned} \quad (2.23)$$

This implies $\sum p_k < \infty$, which completes the proof of (2.20). Next, we prove that

$$\liminf_{n \to \infty} \Delta_n^+(a_n)/\log n \geqslant c\beta_c^+ \qquad \text{a.s.} \tag{2.24}$$

for $a_n = (c \log n)/n$. Again, by the Borel–Cantelli lemma, it suffices to show that $\sum p_n < \infty$, where now $p_n = P^*(\Delta_n^+(a_n) < (1-\varepsilon)\, c\beta_c^+ \log n)$. From (2.16) we have

$$p_n \leqslant \exp\{-K_{\mathbf{D}}(a_n)\, P(B(n, a_n) \geqslant (1-\varepsilon)\, c\beta_c^+ \log n)\} \tag{2.25}$$

and from (2.18), we have for large n, since $h\uparrow$,

$$\begin{aligned}
\log P(B(n, a_n) &\geqslant (1-\varepsilon)\, c\beta_c^+ \log n) \\
&= \{(1-\varepsilon)\, c\beta_c^+ \log n\}\{\log(1/((1-\varepsilon)\,\beta_c^+)) \\
&\quad - 1/((1-\varepsilon)\,\beta_c^+) + 1 + o(1)\} \\
&= -c \log n\{h((1-\varepsilon)\,\beta_c^+) + o(1)\} \geqslant -(1-\delta) \log n,
\end{aligned} \tag{2.26}$$

for some $\delta > 0$. Combining (2.25) with (2.26) yields

$$p_n \leqslant \exp(-K_{\mathbf{D}}(a_n)\, n^{-(1-\delta)}) := \exp(-r_n). \tag{2.27}$$

Using (2.5), we obtain (cf. (2.23))

$$\begin{aligned}
\liminf_{n \to \infty} \frac{\log r_n}{\log n} &= \left(\liminf_{n \to \infty} \frac{\log K_{\mathbf{D}}(a_n)}{\log(1/a_n)}\right) - (1-\delta) \\
&\geqslant 1 - (1-\delta) = \delta,
\end{aligned} \tag{2.28}$$

which implies that $\sum p_n < \infty$. Hence (2.6) is proved for Δ_n^+.

We now consider (2.8). Similar to (2.20), we first prove that under the assumptions of part (II)

$$\limsup_{n \to \infty} \log(1/c_n)\, \Delta_n^+(a_n)/\log n \leqslant 1 \qquad \text{a.s.} \tag{2.29}$$

Let $\varepsilon > 0$ and $n_k = 2^k$ for $k \in \mathbb{N}$. Since $\log(1/c_n)/\log n\downarrow$ and $a_n\downarrow$ we have

$$\max_{n_{k-1} < n \leqslant n_k} \frac{\log(1/c_n)}{\log n} \Delta_n^+(a_n) \leqslant \frac{\log(1/c_{n_{k-1}})}{\log n_{k-1}} \Delta_{n_k}^+(a_{n_{k-1}}). \tag{2.30}$$

Hence, it suffices to show that $\sum p_k < \infty$, with

$$p_k = P^*(\Delta_{n_k}^+(a_{n_{k-1}}) \geqslant (1+\varepsilon) \log n_{k-1}/\log(1/c_{n_{k-1}})).$$

Writing

$$\lambda_k = \frac{\log n_{k-1}}{n_k a_{n_{k-1}}} \log(1/c_{n_{k-1}})$$

and using (2.14) we have for large k

$$\begin{aligned} p_k &= P^*(\Delta^+_{n_k}(a_{n_{k-1}}) \geqslant (1+\varepsilon)\, n_k a_{n_{k-1}} \lambda_k) \\ &\leqslant M_{\mathbf{D}}(a_{n_{k-1}}, \varepsilon) \exp(-n_k a_{n_{k-1}}(1+\varepsilon)\, h(\lambda_k)). \end{aligned} \tag{2.31}$$

Since $c_n \to 0$ it is easily seen that $\lambda_k \to \infty$ as $k \to \infty$. We also have $h(x) \sim x \log x$ as $x \to \infty$ and $\log \lambda_k / \log(1/c_{n_{k-1}}) \to 1$ as $k \to \infty$. Hence, from (2.31), we have for large k

$$\begin{aligned} p_k &\leqslant M_{\mathbf{D}}(a_{n_{k-1}}, \varepsilon) \exp\{-n_k a_{n_{k-1}}(1+\tfrac{1}{2}\varepsilon)\, \lambda_k \log \lambda_k\} \\ &\leqslant M_{\mathbf{D}}(a_{n_{k-1}}, \varepsilon) \exp\{-(1+\tfrac{1}{4}\varepsilon) \log n_{k-1}\} \\ &= M_{\mathbf{D}}(a_{n_{k-1}}, \varepsilon)\, n_{k-1}^{-(1+(1/4)\varepsilon)}. \end{aligned} \tag{2.32}$$

But now we are in the same situation as at the end of (2.22). Hence, since again $\log(1/a_n)/\log n \to 1$ as $n \to \infty$,

$$\limsup_{k \to \infty} \frac{\log p_k}{\log n_{k-1}} \leqslant -\frac{1}{4}\varepsilon. \tag{2.33}$$

This completes the proof of (2.29).

To complete the proof of (2.8), we need to show that under the assumptions of part (II)

$$\liminf_{n \to \infty} \log(1/c_n)\, \Delta^+_n(a_n)/\log n \geqslant 1 \qquad \text{a.s.} \tag{2.34}$$

As in the proof of (2.24), it suffices to show $\sum p_n < \infty$, where now

$$p_n = P^*(\Delta^+_n(a_n) < (1-\varepsilon)(\log n)/\log(1/c_n)).$$

Using (2.18), we see that for large n (cf. (2.26))

$$\begin{aligned} &\log P(B(n, a_n) \\ &\quad \geqslant (1-\varepsilon)(\log n)/\log(1/c_n)) \\ &\quad = (1-\varepsilon) \frac{\log n}{\log(1/c_n)} \left\{ \log\left(\frac{c_n \log(1/c_n)}{1-\varepsilon} \right) - \frac{c_n \log(1/c_n)}{1-\varepsilon} + 1 + o(1) \right\} \\ &\quad = -(1-\varepsilon) \log n \left\{ 1 - \frac{\log(\log(1/c_n)/(1-\varepsilon)) + 1 + o(1)}{\log(1/c_n)} \right\} \\ &\quad = -(1-\varepsilon) \log n \{1 + o(1)\} \geqslant -(1-\delta) \log n, \end{aligned} \tag{2.35}$$

for some $\delta > 0$. Combining (2.35) with (2.16) it readily follows that

$$p_n \leqslant \exp(-K_{\mathbf{D}}(a_n)\, n^{-(1-\delta)}).$$

Now the proof of (2.34) can be completed as in (2.28). ∎

3. Proofs of Theorems 1–3

The proofs of Theorems 1–3 hinge on the following simple event identities (cf. Mason [20, Lemma 1]).

Event Identities. *For $n \in \mathbb{N}$, $1 \leqslant k < n$, and "small" $a > 0$, we have*

$$\{M_{k,n} > a\} = \{\Delta_n^-(a) < k\} \tag{3.1}$$

and

$$\{m_{k,n} < a\} = \{\Delta_n^+(a) > k\}. \tag{3.2}$$

Proof. Since the proofs of (3.1) and (3.2) are almost identical, we confine ourselves to a proof of (3.1). First, suppose $M_{k,n} > a$, then because of (1.2) there exists a $C \in \mathbf{C}$ with $nP_n(C) < k$ and $\lambda(C) > a$. By (C.3), we can choose a $C' \in \mathbf{C}_a$ such that $C' \subset C$, which implies that $\Delta_n^-(a) < k$. Now suppose $\Delta_n^-(a) < k$. By (C.4), there exists a $C' \in \mathbf{C}$ with $\lambda(C') > a$ and $nP_n(C') < k$, which implies that $M_{k,n} > a$. ∎

Proof of Theorem 1. (I). Observe that we may assume w.l.o.g. that $k_n = [\lambda \log n]^+$ since $M_{k,n}\uparrow$ as $k\uparrow$. Choose $\lambda \in (0, \infty)$ and set $c\beta_c^- = \lambda$. Note that $1/\beta_c^- = \alpha_\lambda^+$ and $c = \lambda\alpha_\lambda^+$. Let $0 < \varepsilon < 1$. From (3.1), we have

$$\begin{aligned}&\left\{\Delta_n^-\left(\frac{(1+\varepsilon)c\log n}{n}\right) < c\beta_c^- \log n \quad \text{i.o.}\right\}\\ &\qquad = \{nM_{k_n,n} > (1+\varepsilon)c\log n \quad \text{i.o.}\}\end{aligned} \tag{3.3}$$

and

$$\begin{aligned}&\left\{\Delta_n^-\left(\frac{(1-\varepsilon)c\log n}{n}\right) \geqslant c\beta_c^- \log n \quad \text{i.o.}\right\}\\ &\qquad = \{nM_{k_n,n} \leqslant (1-\varepsilon)c\log n \quad \text{i.o.}\}.\end{aligned} \tag{3.4}$$

It is easily seen from Theorem 4 and its proof that condition (2.3) is only needed for the results for Δ_n^+ and that condition (2.4) is only needed for the results for Δ_n^-. Applying Theorem 4, Part I, we have for either choice of sign

$$\Delta_n^-\left(\frac{(1\pm\varepsilon)c\log n}{n}\right)\Big/\log n \to c(1\pm\varepsilon)\,\beta_{(1\pm\varepsilon)c}^- \qquad \text{a.s.} \tag{3.5}$$

Note that $c(1-\varepsilon)\,\beta^-_{(1-\varepsilon)c} < c\beta^-_c < c(1+\varepsilon)\,\beta^-_{(1+\varepsilon)c}$. Hence, from (3.3) and (3.4), we have

$$P^*(nM_{k_n,n} > (1+\varepsilon)c\log n \quad \text{i.o.}) = 0; \tag{3.6}$$

$$P^*(nM_{k_n,n} \leqslant (1-\varepsilon)c\log n \quad \text{i.o.}) = 0. \tag{3.7}$$

(II) Since $M_{k,n}\uparrow$ as $k\uparrow$ and $\lambda\alpha^+_\lambda \downarrow 1$ as $\lambda \downarrow 0$, we immediately have from (1.9) that

$$\limsup_{n\to\infty} \frac{nM_{k_n,n}}{\log n} \leqslant 1 \qquad \text{a.s.} \tag{3.8}$$

if $k_n/\log n \to 0$. Again, using $M_{k,n}\uparrow$ as $k\uparrow$, we see that to complete the proof of part (II), it suffices to show that

$$\liminf_{n\to\infty} nM_{1,n}/\log n \geqslant 1 \qquad \text{a.s.} \tag{3.9}$$

From (3.1), we have for every $0<\varepsilon<1$

$$\left\{\Delta^-_n\left(\frac{(1-\varepsilon)\log n}{n}\right) \geqslant 1 \quad \text{i.o.}\right\} = \{nM_{1,n} \leqslant (1-\varepsilon)\log n \quad \text{i.o.}\}. \tag{3.10}$$

Now, (2.7) yields (3.9). This completes the proof of Theorem 1. ∎

Proof of Theorem 2. The proof of part (I) is, mutatis mutandis, the same as the proof of part (I) of Theorem 1 and will therefore be omitted.

(II) Define c_n by $l_n = \log n/\log(1/c_n)$. Note that for either choice of sign $c_n^{1\pm\varepsilon} \downarrow 0$ and $\log(1/c_n^{1\pm\varepsilon})/\log n \downarrow 0$. Hence, we can apply Theorem 4, Part (II), which yields for either choice of sign

$$\frac{\log(1/c_n)}{\log n}\,\Delta^+_n\left(\frac{c_n^{1\pm\varepsilon}\log n}{n}\right) \to \frac{1}{1\pm\varepsilon} \qquad \text{a.s.} \tag{3.11}$$

But from (3.2), we have

$$\left\{\Delta^+_n\left(\frac{c_n^{1+\varepsilon}\log n}{n}\right) > l_n \quad \text{i.o.}\right\} = \{nm_{k_n,n} < c_n^{1+\varepsilon}\log n \quad \text{i.o.}\} \tag{3.12}$$

and

$$\left\{\Delta^+_n\left(\frac{c_n^{1-\varepsilon}\log n}{n}\right) \leqslant l_n \quad \text{i.o.}\right\} = \left\{nm_{k_n,n} \geqslant c_n^{1-\varepsilon}\log n \quad \text{i.o.}\right\}. \tag{3.13}$$

Combining (3.11) and (3.12), we see that

$$P^*\left(\frac{l_n}{\log(1/n)}\log\left(\frac{nm_{k_n,n}}{\log n}\right) > 1+\varepsilon \quad \text{i.o.}\right)$$
$$= P^*(nm_{k_n,n} < c_n^{1+\varepsilon}\log n \quad \text{i.o.}) = 0 \tag{3.14}$$

and similarly, we have from (3.11) and (3.13)

$$P^*\left(\frac{l_n}{\log(1/n)}\log\left(\frac{nm_{k_n,n}}{\log n}\right) \leqslant 1-\varepsilon \quad \text{i.o.}\right) = 0. \tag{3.15}$$

Combining (3.14) and (3.15) with the fact that $k_n/l_n = [l_n]/l_n \to 1$ as $n \to \infty$ yields (1.12).

(III) Let $k \in \mathbb{N}$ be fixed and choose $\varepsilon > 0$. Observe that for small enough $\varepsilon > 0$,

$$\left[\left(1+\frac{1}{k}+\varepsilon\right)\Big/\left(\frac{1}{k}+\varepsilon\right)\right] = k$$

and

$$\left[\left(1+\frac{1}{k}-\varepsilon\right)\Big/\left(\frac{1}{k}-\varepsilon\right)\right] = k+1.$$

Hence, we have from Theorem 4, Part (III),

$$\Delta_n^+\left(\frac{1}{n^{1+(1/k)+\varepsilon}}\right) \to k \qquad \text{a.s.} \tag{3.16}$$

and

$$\Delta_n^+\left(\frac{1}{n^{1+(1/k)-\varepsilon}}\right) \to k+1 \qquad \text{a.s.} \tag{3.17}$$

But from (3.2), we have

$$\left\{\Delta_n^+\left(\frac{1}{n^{1+(1/k)+\varepsilon}}\right) > k \quad \text{i.o.}\right\} = \left\{m_{k,n} < \frac{1}{n^{1+(1/k)+\varepsilon}} \quad \text{i.o.}\right\} \tag{3.18}$$

and

$$\left\{\Delta_n^+\left(\frac{1}{n^{1+(1/k)-\varepsilon}}\right) \leqslant k \quad \text{i.o.}\right\} = \left\{m_{k,n} \geqslant \frac{1}{n^{1+(1/k)-\varepsilon}} \quad \text{i.o.}\right\}. \tag{3.19}$$

Combining (3.16) and (3.18), we see that

$$P^*\left(\frac{\log m_{k,n}}{\log(1/n)} > 1+\frac{1}{k}+\varepsilon \quad \text{i.o.}\right) = P^*\left(m_{k,n} < \frac{1}{n^{1+(1/k)+\varepsilon}} \quad \text{i.o.}\right) = 0 \tag{3.20}$$

and similarly, we have from (3.17) and (3.19)

$$P^*\left(\frac{\log m_{k,n}}{\log(1/n)} \leqslant 1+\frac{1}{k}-\varepsilon \quad \text{i.o.}\right)=0. \tag{3.21}$$

From (3.20) and (3.21) we get (1.13), which completes the proof of Theorem 2. ∎

Before we present the proof of Theorem 3, we require some additional facts.

INEQUALITY 3. *Let $n \in \mathbb{N}$, $k \in \mathbb{N}_0$, $p \in (0,1)$, and $k < np$. Then, we have*

$$P(B(n,p) \leqslant k) \leqslant e^{-np}\frac{(np)^k}{k!}e^{pk}\frac{1}{1-k/(np)}. \tag{3.22}$$

Proof. We have, writing $e^{-np}(np)^k e^{pk}/k! = R$,

$$\begin{aligned} R^{-1}\sum_{j=0}^{k}\binom{n}{j}p^j(1-p)^{n-j} &\leqslant R^{-1}\sum_{j=0}^{k}e^{-p(n-j)}(np)^j/j! \\ &= \sum_{j=0}^{k}\frac{k!}{j!\,e^{p(k-j)}(np)^{k-j}} \leqslant \sum_{j=0}^{k}(k/(np))^{k-j} \\ &= \sum_{j=0}^{k}(k/(np))^j \leqslant (1-k/(np))^{-1}. \quad ∎ \end{aligned} \tag{3.23}$$

FACT 4 (Devroye [11]). Let $n, k \in \mathbb{N}$, $1 \leqslant k \leqslant n$, and $p \in (0,1)$. Then we have

$$P(B(n,p) < k) \geqslant e^{-np}\frac{(np)^{k-1}}{(k-1)!}\exp\left(\frac{-np^2}{2(1-p)}\right)\left(1-\frac{k}{n}\right)^{k-1}. \tag{3.24}$$

FACT 5 (Deheuvels and Devroye [8]). Let $\{m_n\}_{n=1}^{\infty}$ be a sequence of numbers with $m_n \to \infty$ and $m_n/\log n \to 0$ as $n \to \infty$. Then

$$m_n \log((\log n)/m_n) = o(\log n) \tag{3.25}$$

and

$$\log\log n = o(m_n \log((\log n)/m_n)). \tag{3.26}$$

Proof of Theorem 3. We first prove that

$$\limsup_{n\to\infty}\frac{nM_{k_n,n}-\log n}{k_n(\log((\log n)/k_n)}\leqslant 1 \qquad \text{a.s.} \tag{3.27}$$

Let $\varepsilon > 0$ and write $n_k = [e^{\sqrt{k}}]$, for $k \in \mathbb{N}$. We have to show that

$$P^*\left(nM_{k_n, n} - \log n > (1+2\varepsilon)\, k_n \log\left(\frac{\log n}{k_n}\right) \quad \text{i.o.}\right) = 0.$$

Observe that

$$\max_{n_{k-1} < n \leqslant n_k} M_{k_n, n} \leqslant M_{k_{n_k}, n_{k-1}}. \tag{3.28}$$

Hence by the Borel–Cantelli lemma, it suffices to show that $\sum p_k < \infty$, where

$$p_k = P^*\left(M_{k_{n_k}, n_{k-1}} > \min_{n_{k-1} < n \leqslant n_k} \left(\frac{\log n}{n} + (1+2\varepsilon)\frac{k_n}{n} \log\left(\frac{\log n}{k_n}\right)\right)\right). \tag{3.29}$$

Using the assumptions on $\{l_n\}_{n=1}^{\infty}$, it is easily seen that the minimum in this expression is larger than

$$\frac{\log n_k}{n_k} + (1+\varepsilon)\frac{l_{n_k}}{n_k} \log\left(\frac{\log n_k}{l_{n_k}}\right), \tag{3.30}$$

for large k. Write $b_n = l_n \log((\log n)/l_n)$ and $a_n = n^{-1}(\log n + (1+\varepsilon)\, b_n)$. Then combining (3.29) and (3.30) and using (3.1), we obtain for large k,

$$p_k \leqslant P^*(M_{k_{n_k}, n_{k-1}} > a_{n_k}) = P^*(\Delta^-_{n_{k-1}}(a_{n_k}) < k_{n_k}). \tag{3.31}$$

Now choosing $v = v_k = (n_{k-1} a_{n_k})^{-1}$ and using the definition of $N_C(a, v)$, we see that (cf. Inequality 1)

$$P^*(\Delta^-_{n_{k-1}}(a_{n_k}) < k_{n_k}) \leqslant N_C(a_{n_k}, v_k)\, P(B(n_{k-1}, a_{n_k}(1-v_k)) < k_{n_k}). \tag{3.32}$$

Now, our main task is evaluating $r_k := P(B(n_{k-1}, a_{n_k}(1-v_k)) < k_{n_k})$. Using Inequality 3, we have

$$r_k \leqslant \frac{e^{-n_{k-1} a_{n_k}(1-v_k)} (n_{k-1} a_{n_k}(1-v_k))^{k_{n_k}}}{k_{n_k}!} \times e^{a_{n_k}(1-v_k) k_{n_k}} \left(1 - \frac{k_{n_k}}{n_{k-1} a_{n_k}(1-v_k)}\right)^{-1}. \tag{3.33}$$

Observe that $a_{n_k}(1-v_k)\, k_{n_k} \to 0$ and $k_{n_k}/(n_{k-1} a_{n_k}(1-v_k)) \to 0$ as $k \to \infty$. Hence

$$e^{a_{n_k}(1-v_k) k_{n_k}} \left(1 - \frac{k_{n_k}}{n_{k-1} a_{n_k}(1-v_k)}\right)^{-1} = 1 + o(1). \tag{3.34}$$

From (3.33), (3.34), and Stirling's formula ($k! \geqslant k^k/e^k$), we have

$$r_k \leqslant (1+o(1)) \exp\{-n_{k-1} a_{n_k} + n_{k-1} a_{n_k} v_k + k_{n_k} \log(n_{k-1} a_{n_k}) + k_{n_k} \log(1 - v_k) - k_{n_k} \log k_{n_k} + k_{n_k}\}. \tag{3.35}$$

Using $n_{k-1}/n_k = 1 - (1+o(1))/(2 \log n_k)$ as $k \to \infty$, $k_n = [l_n]^+$, and Fact 5, it is readily seen that the right side of (3.35) is for large k less than

$$(1+o(1))\, n_k^{-1} \exp\left(-\frac{1}{2} \varepsilon l_{n_k} \log\left(\frac{\log n_k}{l_{n_k}}\right)\right) \leqslant n_k^{-1} \exp(-C \log\log n_k) = n_k^{-1} (\log n_k)^{-C}, \tag{3.36}$$

for any $C > 0$.

By condition (1.14), there exists an $M > 0$ such that for all a, $v > 0$ "sufficiently small"

$$N_{\mathbf{C}}(a, v) \leqslant a^{-1} v^{-M} (\log(1/a))^M. \tag{3.37}$$

Combining (3.31)–(3.32) and (3.35)–(3.37) and using $v_k = (1+o(1))/\log n_k$, we see that for large k,

$$p_k \leqslant (1+o(1)) \frac{n_k}{\log n_k} (\log n_k)^M (\log n_k)^M n_k^{-1} (\log n_k)^{-C} = (1+o(1))(1/\log n_k)^{C+1-2M}. \tag{3.38}$$

Using $n_k \geqslant e^{\sqrt{k^2}}$ and choosing $C > 2M + 1$, we see that $\sum p_k < \infty$, which completes the proof of (3.27).

Finally, we have to show that

$$\liminf_{n \to \infty} \frac{n M_{k_n, n} - \log n}{k_n \log((\log n)/k_n)} \geqslant 1 \qquad \text{a.s.} \tag{3.39}$$

or, equivalently,

$$P^*\left(n M_{k_n, n} - \log n \leqslant (1-\varepsilon)\, k_n \log\left(\frac{\log n}{k_n}\right) \quad \text{i.o.}\right) = 0, \tag{3.40}$$

where $0 < \varepsilon < 1$ is arbitrary. So by the Borel–Cantelli lemma, it suffices to prove that $\sum p_n < \infty$, where $p_n = P^*(n M_{k_n, n} - \log n \leqslant (1-\varepsilon)\, b_n)$ and $b_n = k_n \log((\log n)/k_n)$. Writing $a_n = n^{-1}(\log n + (1-\varepsilon)\, b_n)$, we have from (3.1) that for large n

$$p_n = P^*(\Delta_n^-(a_n) \geqslant k_n) \tag{3.41}$$

and by (a slight modification of) (2.17), we see that for large n

$$P^*(\Delta_n^-(a_n) \geqslant k_n) \leqslant \exp\{-K_C(a_n)\, P(B(n, a_n) < k_n)\}. \tag{3.42}$$

Similarly, as below (3.32), we now have to evaluate $r_n := P(B(n, a_n) < k_n)$. Using Fact 4, we see that

$$r_n \geqslant \frac{e^{-na_n}(na_n)^{k_n-1}}{(k_n-1)!} \exp\left(\frac{-na_n^2}{2(1-a_n)}\right)\left(1-\frac{k_n}{n}\right)^{k_n-1}. \tag{3.43}$$

Observe that since $k_n = o(\log n)$

$$\left(1-\frac{k_n}{n}\right)^{k_n-1} \geqslant 1-\frac{k_n^2}{n} = 1+o(1) \tag{3.44}$$

and $na_n^2/(1-a_n) = o(1)$. Hence, we have for large n

$$r_n \geqslant (1+o(1)) \exp\{-na_n + (k_n-1)\log na_n - \log((k_n-1)!)\}. \tag{3.45}$$

Using Stirling's formula $((k-1)! \leqslant 3k^k/e^k)$, we have from (3.45) for large n

$$r_n \geqslant (1+o(1)) \exp\{-na_n + (k_n-1)\log na_n - k_n \log k_n + k_n - \log 3\}. \tag{3.46}$$

Now by some elementary analysis, using Fact 5, we see that the right side of (3.46) is larger than (for large n)

$$(1+o(1))\, n^{-1} \exp(\tfrac{1}{2}\varepsilon b_n) \geqslant n^{-1} \exp(C \log\log n) = n^{-1}(\log n)^C, \tag{3.47}$$

for any $C > 0$.

From condition (1.15), we have that there exists an $M > 0$ such that for all $a > 0$ "sufficiently small"

$$K_C(a) \geqslant a^{-1}(\log(1/a))^{-M}. \tag{3.48}$$

Combining (3.41)–(3.42) and (3.46)–(3.48), we see that for large n

$$\begin{aligned} p_n &\leqslant \exp\{-(1+o(1))\, n(\log n)^{-1}(\log n)^{-M} n^{-1}(\log n)^C\} \\ &= \exp\{-(1+o(1))(\log n)^{C-M-1}\}. \end{aligned} \tag{3.49}$$

By choosing $C > M+2$, we obtain $\sum p_n < \infty$, which completes the proof of (3.39) and hence the proof of Theorem 3. ∎

APPENDIX

In order to carry out, in an economical way, the necessary calculations which show that the examples introduced in Section 1 actually satisfy the required "entropy" conditions, we make a number of observations. It is easily seen that (1.14) implies (1.7) and that (1.15) implies (1.8). Therefore, it suffices to prove that our examples satisfy the following four properties:

(A) *for all* $\varepsilon > 0$ *there exists an* $a' > 0$ *such that for all* $0 < a \leqslant a'$

$$M_{\mathbf{D}}(a, v) \leqslant (1/a)^{1+\varepsilon};$$

(B) *for all* $\varepsilon > 0$, *there exists an* $a' > 0$ *such that for all* $0 < a \leqslant a'$

$$K_{\mathbf{D}}(a) \geqslant (1/a)^{1-\varepsilon};$$

(C) *there exist* M, a', $v' > 0$ *such that for all* $0 < a \leqslant a'$ *and* $0 < v \leqslant v'$

$$N_{\mathbf{C}}(a, v) \leqslant (1/a)\{(1/v)\log(1/a)\}^{M};$$

(D) *there exist* M, $a' > 0$ *such that for all* $0 < a \leqslant a'$

$$K_{\mathbf{C}}(a) \geqslant (1/a)(\log(1/a))^{-M}.$$

Observe that D implies B if $\mathbf{D}$ is related to $\mathbf{C}$ as in all our examples. Therefore, we only have to show (A), (C), and (D) for (multivariate) squares, rectangles, and circles. Since it is obvious that (A) and (C) are true for squares if they are true for rectangles and that (D) is true for rectangles, if it is true for squares, it is enough to prove (A) and (C) for rectangles and circles and (D) for squares and circles.

We begin by demonstrating property (A) for (open) rectangles. From Einmahl [12, pp. 68–70], it is immediate that

$$M_D(a, v) \leqslant \binom{l-1}{d-1}\left(\frac{\theta}{1-\theta}\right)^d \frac{1}{a}, \tag{A.1}$$

where $l \leqslant \log(1/a)/\log(1/\theta)$ and $\theta = (1/(1+v))^{1/(2d+1)}$. (Here and in the remainder of the Appendix, it is tacitly assumed that a and v are "sufficiently small.") The right side of (A.1) is in turn less than or equal to

$$\left(\frac{\log(1/a)}{\log(1/\theta)}\right)^{d-1}\left(\frac{\theta}{1-\theta}\right)^d \frac{1}{a} \leqslant 2\frac{1}{a}\left(\log\frac{1}{a}\right)^{d-1}\left(\frac{1}{c_1 v}\right)^{d-1}\left(\frac{1}{c_1 v}\right)^d$$
$$\leqslant c_2 \frac{1}{a}\left(\log\frac{1}{a}\right)^{d-1}\left(\frac{1}{v}\right)^{2d-1}, \tag{A.2}$$

where $c_1, c_2 \in (0, \infty)$ only depend on the dimension d. Of course, (A.1) and (A.2) together imply that condition (A) is satisfied. In an almost identical manner, it can be shown that the right side of inequality (A.2) is also upper bound for $N_{\mathbf{C}}(a, v)$ if $\mathbf{C}$ is the class of closed rectangles. This means that property C is satisfied, provided we choose $M > 2d-1$.

Next, we show that property (A) holds for (open) circles. Assume we have a circle τ in $\mathbb{R}^d$ with $\lambda(\tau) = (v/(d+1))^d a$. Then all circles with center in τ and Lebesgue measure equal to a are contained in the circle with the same center *as* τ and with Lebesgue measure $(1+v)a$. Observe that there exists a square S, contained in τ, which has sides parallel to the coordinate axes and for which $\lambda(S) = cv^d a$, for some $c \in (0, 1)$ depending only on d. Hence, we immediately have that

$$M_D(a, v) \leqslant 2/(cv^d a), \tag{A.3}$$

which proves property (A) for circles. Again, a similar proof shows that the same number is an upper bound for $N_C(a, v)$ if C is the class of closed circles. Choosing $M > d$ proves property (C) for this case.

Finally, we show property (D) for (closed) squares and circles. Let us begin with the squares. It is easily seen that there exists a collection of $[1/a^{1/d}]^d$ closed squares with disjoint interiors and Lebesgue measure a. But $[1/a^{1/d}]^d \geqslant 1/(2a)$, which proves property (D). Since for fixed dimension d, there is a fixed ratio between a square and the largest circle contained in it, it is readily seen that we have for closed circles

$$K_C(a) \geqslant c/a, \tag{A.4}$$

where $c \in (0, 1)$ depends only on d. Hence, the proof that all the examples indeed satisfy the "entropy" conditions in the theorems is completed.

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